Minimum Time Validation for Hybrid Task Planning

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Abstract

The problem of mixed discrete-continuous task planning for mechanical systems, such as aerial drones or other autonomous units, can often be treated as a sequence of point-to-point trajectories. The minimum time optimal solution between points in the plan is critical not only for the calculation of the trajectory in cases where the goal has to be achieved quickly but also for the feasibility checking of the plan and the planning process itself, especially in the presence of deadlines and temporal constraints. In this paper, we address the minimum time problem for a second-order system with quadratic drag, under state (velocity) and control (acceleration) constraints. Closed-form expressions for the trajectory are derived and the optimality is proven using the Pontryagin Maximum Principle. Simulations supporting the results are provided and compared with those of an open source academic optimal control solver.

Introduction

Robots operating in the real world often have to come up with a sequence of actions, i.e., a plan, which will take them from an initial state to the desired goal state. During planning the robots have to take into account both discrete and continuous changes, as well as temporal constraints. Hybrid temporal planners such as Scotty (Fernandez-Gonzalez, Williams, and Karpas 2018), COLIN (Coles et al. 2012) and Kongming (Li and Williams 2008) have been introduced in the last few years, and managed to deal with such mixed discrete-continuous temporal planning problems quite well.

The Scotty planner (Fernandez-Gonzalez, Williams, and Karpas 2018) addressed the problem of finding a discrete temporal plan with a continuous control policy for systems represented by a first order integrator. Scotty supports durative actions with a flexible (controllable) duration. It combines search through plan skeletons with trajectory optimization via mathematical programming. For the search phase, Scotty uses non-admissible heuristics which may result in a non-optimal action sequence, but the continuous assignments are done according to a global objective. Thus, the output plan is optimal under the model assumptions and the skeleton imposed by the search. The first order dynamics assumption makes it possible to formulate a solvable Second Order Cone Programming (SOCP) optimization problem, but no requirement on the continuity of the input (velocity) is imposed. Thus, Scotty can output a trajectory which is infeasible for a physical system, e.g., where the velocity changes significantly within a very small (ϵ) amount of time.

Our long-term objective is to make planners like Scotty aware of more complex dynamic constraints in order to find a more realistic trajectory, while still keeping the simplicity of planning with a first order integrator. Scotty calculates a linear trajectory from an initial condition to a goal condition for each temporal action in the plan, composing a piecewise linear trajectory. These conditions are imposed by the actions' start and end conditions. As previously mentioned, Scotty formulates this problem as an SOCP optimization problem, and thus cannot reflect real higher order physical dynamics. In previous work (Taitler et al. 2019b), we presented an analytical method without an explicit solution for trajectory optimization of a second-order integrator system with quadratic drag, optimizing a mixed time-energy performance criterion. The segment trajectory optimization is performed after completion of Scotty's planning in the first order model, and can identify segments of the plan that are infeasible according to the second order model. A method to resolve the infeasibility is not proposed.

In this paper, we address the physical feasibility problem inspired Scotty, i.e. determining if the change in the continues variables between two following events is physically achievable. We address this as an optimal control problem with the same model and constraints as our previous work (Taitler et al. 2019b), but with a pure minimum time optimization criterion. We emphasise that the minimum time objective is a tool to bound the physical requirement of two following events, and not a global objective for the planner. We show it is possible to derive *explicit* closed-form expressions for the optimal time, and the velocity and control profiles. This has a direct impact on the optimization problem of the search stage. The expression for the minimum time between two points (position and velocity) in the continuous state space is a tight lower-bound that has to be satisfied by *any* feasible plan produced by the planner. The minimum time expression can be used as a post-search validation for the time requirement, but it can also be used as a constraint in the optimization problem that is being solved during the search.

In order to derive these lower-bound expressions we rely on tools from optimal control theory. The general problem of optimal control has been studied extensively, and solid mathematical tools have been established by (Pontryagin et al. 1962), and extended to problems with state constraints (Hartl, Sethi, and Vickson 1995). Minimum time problems are of particular interest, especially for trajectory planning problems, and many approaches have been established (Ata 2007). In (Berger, Ioslovich, and Gutman 2015) a solution for second order and third order models with control and state constraints for a time-optimal criterion, with drag proportional to the sign of the velocity was presented. An SOCP approach for combined criteria was introduced in (Verscheure et al. 2008) for dynamics with a quadratic term without path constraints. A numerical toolbox for solving optimal control problems which was developed at the Institute of Flight System Dynamics at TUM is FALCON (Rieck et al. 2016), which, by using the non-linear interior point optimizer IPOPT (Wächter and Biegler 2006), solves complex optimal control problems. FALCON will be used for comparison of the results in this work. The solution presented here for the minimum time criterion for a second-order system with quadratic drag and velocity constraints is given here explicitly, and in closed form, based on the parameters of the problem. As far as the authors know, this explicit solution has not been published previously.

An overview of the whole planning approach, and where this work fits is provided in the next section.

Background

We now review some necessary background.

Planning Formalism

We consider planning problems formulated in a language similar to PDDL-S (Fernandez-Gonzalez, Williams, and Karpas 2018), the language used as input for the Scotty planner. However, we assume our system is a second-order system, and thus we directly control accelerations rather than velocities, which requires some changes we highlight below.

We address a hybrid (mixed continuous-discrete) planning task, which is given by the tuple $\langle P, X, A, I, G, C, J \rangle$, where

- P is a set of propositions, which define the discrete component of the task.
- X is a set of real-valued variables, which define the continuous component of the task. Since our system is a second-order system, we also keep track of the velocities (first derivatives) of each state variables, denoted $\dot{X} = \{\dot{x}_1, ..., \dot{x}_n\}$ of dotted variables.
- A is the set of hybrid durative activities.
  Each activity a is associated with two time points: its start and its end. Each time point can have both discrete and continuous conditions (which must hold at that exact time point), and discrete effects.
  Additionally, each action has an invariant condition which must hold throughout its execution, and continuous effects, which continuously modify the values of the continuous variables X throughout the execution of the action.
  Like Scotty (Fernandez-Gonzalez, Williams, and Karpas 2018), these continuous effects are controlled by control variables (U, below). However, unlike Scotty, we do not control the velocities directly, but rather the accelerations.
- I is the initial state , which is a complete assignment to the propositional variables $p_0 = p(0)$ and state variables $x_0 = x(0)$ at the beginning, $I = \langle p_0, x_0 \rangle$.
- G is the goal set, which is a partial assignment to the propositional variables P_G whose value needs to be true at the end of the plan, and a set of state constraints X_G , which have to be satisfied at the end of the plan, $G = \langle P_G, X_G \rangle$.
- C is the tuple $\langle U, U_C \rangle$, where U is the vector of control variables (accelerations), and U_C is the set of constraints operating on the control variables.
- J is the objective function.

The solution to the hybrid planning task is the schedule of durative activities, and the trajectory and control signal of the system during each activity. This solution is referred to in the frame of this work as the *hybrid grounded plan* for the system.

Definition 1 (hybrid grounded plan)**.** *A hybrid grounded plan is a tuple $\mathcal{T}, u_t : \mathbb{R} \to \mathbb{R}^m$, where*

- *$\mathcal{T}$ is the activity schedule, which denotes when each activity should start. $\mathcal{T}$ is given by a list of triplets $\langle a, \tau_s, d \rangle$ where $a \in A$ is an activity, τ_s is the activity's start time, and d is the activity's duration. The makespan of the plan is denoted by T.*
- *$u_t : [0, T] \to \mathbb{R}^m$ is the control trajectory, which assigns values to all the inputs (control variables) at every time point t between 0 and T.*

A valid hybrid grounded plan is a hybrid plan which reaches a goal state from the initial state of the problem while satisfying all the constraints of the problem, defined by the bounds on the continuous components of the system, and the preconditions of the durative activities.

In this context we also define the plan skeleton and the event which comprise it

Definition 2 (Event)**.** *An event is the term used to describe the switch between control modes. Events are the time happenings of the beginnings and ends of the discrete actions, the durative activities.*

Definition 3 (Hybrid plan skeleton)**.** *The plan skeleton S is the ordered list of events $S = (e_0, e_1, ...e_n)$, which are the start and end happenings of the durative actions,*

Note that the plan skeleton is merely the order of events without the assignment of all the continuous variables (including timers), which are not concurrent, and separated by at least an ϵ time constant (Fox and Long 2003). A plan skeleton and assignment of all the continuous variables and durations (timers) is a hybrid grounded plan.

Hybrid planners use some model to express the continuous part of the problem, the most common usage is with a linear rate of change which in control terms is the equivalent to a single integrator (Fernandez-Gonzalez, Williams, and Karpas 2018; Coles et al. 2012). Thus between the j and $j+1$ events the continuous variables evolve according to

$$\dot{x}_j = c_j$$

which is the linear change described by $x_{j+1} = x_j + c_j(t_{j+1} - t_j)$. The constant c_j is the assigned velocity and is also bounded $c_l \leq c_j \leq c_u$.

In this paper, on the other hand, we assume that the system can be described sufficiently well by a *second order* integrator with drag, which is more realistic since it imposes velocity continuity, and flexible enough to bound more complex behaviors due to the drag. Constraints on the velocity (a state) and the input (force) are also imposed on the model,

$$\begin{aligned}
&\dot{x}_1 = x_2 \\
&\dot{x}_2 = u - \frac{1}{2}kx_2^2 \\
&|u| \leq U,\ |x_2| \leq V
\end{aligned}$$

We do not assume that u should be constant between events, it is a general force input. U, V are constraints, and k is the drag coefficient.

Optimal Control

We consider a standard optimal control formulation in continuous time. The goal is to minimize a performance criterion under some system dynamics, and control and state constraints with known starting and final conditions. The optimization problem is of the following general form

$$\begin{aligned}
\underset{u,t_f}{\text{minimize}} \quad & \int_{t_0}^{t_f} l(x,u)dt \\
\text{subject to} \quad & \\
& \dot{x}(t) = f(x,u) \\
& x(t_0) = x_0 \\
& x(t_f) = x_f \\
& C_u u \leq U \qquad t \geq t_0 \\
& Cx \leq X \qquad t \geq t_0.
\end{aligned} \tag{1}$$

Here $x \in \mathbb{R}^n$ is the state vector, $\dot{x} \in \mathbb{R}^n$ is the derivative function of $x(t)$ with respect to time t, and $u \in \mathbb{R}^m$ is the control input to the system, all are functions of time. The function $f(x,u)$ is the system dynamics, and U and X give the component-wise constraints on the control and state vector, respectively.

The Hamiltonian of the system is defined as

$$H_o(x,p,u) = p^T f(x,u) - l(x,u) \tag{2}$$

where $p \in \mathbb{R}^n$ is the vector of co-states. When state vector constraints are present the augmented Hamiltonian is used,

$$H(x,p,u,\lambda) = p^T f(x,u) - l(x,u) - \lambda^T(Cx - X) \tag{3}$$

where $\lambda \geq 0$ is a vector of time-dependent Lagrange multipliers which are non-zero only when the respective state constraint is active. The optimal solution x^*, u^*, p^* must satisfy three conditions. The first is the system dynamics,

$$\frac{dx^*}{dt} = \frac{\partial H(x^*, p^*, u^*, \lambda^*)}{\partial p}. \tag{4}$$

The second is the co-states dynamics,

$$\frac{dp^*}{dt} = -\frac{\partial H(x^*, p^*, u^*, \lambda^*)}{\partial x}, \tag{5}$$

and the third condition is for the control vector,

$$H(x^*, p^*, u^*, \lambda^*) = \max_u H(x^*, p^*, u, \lambda^*). \tag{6}$$

According to the transversality condition for free final time (Kirk 2012)

$$H(t_f) = 0. \tag{7}$$

Finally, when the Hamiltonian is not explicitly time dependent, it holds that $\partial H / \partial t = 0$, (Kirk 2012), and hence the Hamiltonian is constant through the whole process, with the value given in (7),

$$H(t) = 0. \tag{8}$$

The Ricatti Equation

The Riccati equation is a first-order differential equation that is quadratic in the unknown (Bittanti, Laub, and Willems 2012). It is an equation of the form

$$\dot{y}(t) = q_0(t) + q_1(t)y(t) + q_2(t)y^2(t), \tag{9}$$

where $q_0(t) \neq 0$ and $q_2(t) \neq 0$. This non-linear first order ODE can be reduced (Bittanti, Laub, and Willems 2012) to the following linear homogeneous second order ODE,

$$\ddot{\zeta}(t) - R(t)\dot{\zeta}(t) + S(t)\zeta(t) = 0 \tag{10}$$

with $R(t) = q_1(t) + \frac{\dot{q}_2(t)}{q_2(t)}$ and $S(t) = q_0(t)q_2(t)$. The solution of (10) gives the solution of (9) as

$$y(t) = -\frac{\dot{\zeta}(t)}{q_2(t)\zeta(t)}. \tag{11}$$

Statement of the Control Problem

Consider the same system as in (Taitler et al. 2019b), i.e. a second-order integrator with quadratic drag, under acceleration/control constraint and velocity/state constraint,

$$\begin{aligned}
&\dot{x}_1(t) = x_2(t) \\
&\dot{x}_2(t) = u(t) - \frac{1}{2}kx_2^2(t) \\
&|u| \leq U \qquad t \geq t_0 \\
&|x_2| \leq V \qquad t \geq t_0 \\
&x_1(t_0) = x_{10},\ x_2(t_0) = x_{20} \\
&x_1(t_f) = x_{1f},\ x_2(t_f) = x_{2f}.
\end{aligned} \tag{12}$$

Here x_1 $[m]$ is the distance, x_2 $[m/s]$ is the velocity, u $[m/s^2]$ is the acceleration, k is the drag coefficient, U and V are the bounds on the acceleration and the velocity respectively. The initial and final conditions are fixed, and if given from a hybrid plan, the problem is to find a trajectory between two following events.

In (Taitler et al. 2019b) the performance measure was a combination of completion time and consumed energy, whereby the trade-off between the two components in the criterion is tuned with a weight parameter α.

$$\underset{u,t_f}{\text{minimize}} \quad \int_{t_0}^{t_f} \Big(1+\alpha\frac{1}{2}u^2(t)\Big)dt \quad , \quad \alpha>0. \tag{13}$$

In (Taitler et al. 2019a) it was shown analytically that the explicit solution of (12), (13) approaches, for $\alpha > 0$, and $\alpha \to 0$, as a singular limit, the bang-bang structure of the minimum time solution. A comparison was done with the numerically computed minimum time solution. In this paper the analytic solution of the minimum time problem is derived, i.e. for $\alpha = 0$,

$$\underset{u,t_f}{\text{minimize}} \quad \int_{t_0}^{t_f} 1\cdot dt \tag{14}$$

For simplicity, and without loss of generality, we set $t_0 = 0$ in the sequel of the paper.

Optimization

Let us consider the problem (12), (14). We assume that the upper constraint on the velocity might become active and that its lower bound may be omitted. The appropriate Hamiltonian of the problem is then

$$H = p_1x_2 + p_2\Big(u - \frac{1}{2}kx_2^2\Big) - 1 - \lambda_2\big(x_2 - V\big). \tag{15}$$

According to the transversality condition for free final time, we have that

$$H(t) \equiv 0. \tag{16}$$

The time dependent value λ_2 is a Lagrange multiplier for the constraint on the upper bound of the velocity x_2. Note that $\lambda_2 \geq 0$, and may be nonzero only when the velocity constraint is active, i.e., $x_2 = V$. At all other times λ_2 must be zero. The co-states equations are $\dot{p}(t) = -\partial H/\partial x$, specifically,

$$\begin{aligned} &\dot{p}_1(t) = 0 \\ &\dot{p}_2(t) = -p_1(t) + p_2(t)kx_2(t) + \lambda_2(t). \end{aligned} \tag{17}$$

At this point, we make an intelligent assumption on the structure of the solution, inspired by the structure of (Taitler et al. 2019b). It is assumed that at the beginning of the motion, the co-state p_2 is positive and its derivative $\dot{p}_2$ is negative, when the system encounters the velocity bound at time $t = t_1$, p_2 becomes zero, and the Lagrange multiplier λ_2 will ensure that for the duration of the interval $t = [t_1, t_2]$, where t_2 is the time the velocity leaves the bound, $\dot{p}_2$ is also zero. During the last interval, from $t = t_2$ to the end, $x_2 < V$ as the solution moves to the final values, p_2 is negative and its derivative is again negative. If the system does not encounter the velocity bound, then p_2 is always decreasing and $\dot{p}_2$ is negative.

The solution is divided into two cases, distinguished by the status of the constraint on the velocity along the trajectory. The same assumption as in (Taitler et al. 2019b) is used, that the velocity is not reversed during the process and that the velocity constraint satisfies $V \leq \sqrt{2U/k}$, since otherwise the upper-velocity constraint cannot become active with the given control constraint.

Active Upper Bound Velocity Constraint

It is assumed that the constraint on the velocity will be active at some interval along the trajectory, and also that the lower bound on x_2 can be omitted.

Maximization of the Hamiltonian over the control yields

$$\max_u \; p_2u + p_1x_2 - p_2\frac{1}{2}kx_2^2 - 1 - \lambda_2\big(x_2 - V\big). \tag{18}$$

Since u is the maximization parameter, which appears only in one term, it is clear that maximizing H is equivalent to maximizing that term, i.e., $\max_u \; p_2u$. The solution for that is $u(t) = \text{sgn}\big(p_2(t)\big)\cdot U$. Hence when the system accelerates towards the velocity bound, $p_2(t) > 0$, and $u(t) = U$. During the deceleration from the bound to the final condition, $p_2(t) < 0$ and $u(t) = -U$. On the bound the acceleration must be zero, so the control cannot be obtained from the maximization of the Hamiltonian, and the value of $p_2(t)$ must be zero on this singular arc. In order to keep the velocity on the bound V, nature requires that $\dot{x}_2(t) = 0$. For $\dot{x}_2(t) = 0$ to be satisfied, the control must be $u(t) = kV^2/2$. Thus the final velocity profile is a pure Bang-Constant-Bang. Since the control profile is known explicitly now, we can solve the state equations directly.

The velocity differential equation in the acceleration stage under $u(t) = U$ is

$$\begin{cases} \dot{x}_2(t) = U - \frac{1}{2}kx_2^2(t) \\ x_2(0) = x_{20} \end{cases} \tag{19}$$

which is the Riccati equation (9) with $q_0 = U$, $q_1 = 0$, $q_2 = -\frac{1}{2}k$. By solving (10), (11) the solution of (19) is obtained,

$$x_2(t) = \sqrt{\frac{2U}{k}}\frac{\sqrt{2kU}(1-e^{-2\sqrt{2kU}t}) + x_{20}(1+e^{-2\sqrt{2kU}t})}{\sqrt{2kU}(1+e^{-2\sqrt{2kU}t}) + x_{20}(1-e^{-2\sqrt{2kU}t})}. \tag{20}$$

Solving (20) for t with $x_2(t) = V$, yields the time t_1 when the velocity $x_2(t)$ reaches the bound V,

$$t_1 = \frac{1}{\sqrt{2kU}}\ln\Big(\frac{\sqrt{2kU}-x_{20}}{\sqrt{2kU}+x_{20}}\frac{\sqrt{2kU}+Vk}{\sqrt{2kU}-Vk}\Big). \tag{21}$$

To obtain the position at time t_1, (20) should be integrated from zero to time t_1, giving

$$x_1(t_1) = x_{10} + \frac{1}{k}\ln\Big(\frac{2kU-(x_{20}k)^2}{2kU-(Vk)^2}\Big). \tag{22}$$

Here, $\Delta x_1 = \frac{1}{k}\ln\Big(\frac{2kU-(x_{20}k)^2}{2kU-(Vk)^2}\Big)$ is the net displacement in this stage. Note that the time duration of this stage is denoted $\Delta t_1 = t_1$.

Another important point to remark, is that if $V = \sqrt{2U/k}$ then the time where $x_2(t)$ reaches the bound V is $t_1 \to \infty$, so in order for the assumption of three segments trajectory with active upper bound to be valid we must require that $V < \sqrt{2U/k}$.

The deceleration stage starts at $t = t_2$ which is the time where the system leaves the velocity bound V, and the velocity is reduced until reaching the final value, in that stage, the control is $u(t) = -U$. Since there is no explicit dependency on the time t we treat every interval separately for ease of integration, here we set $t_2 = 0$ and the velocity differential equation is

$$\begin{cases} \dot{x}_2(t) = -U - \frac{1}{2}kx_2^2(t) \\ x_2(0) = V \end{cases} \tag{23}$$

which is again the Riccati equation (9) with $q_0 = -U,\ q_1 = 0,\ q_2 = -k/2$. By solving (10), (11) the solution of (23) is obtained,

$$x_2(t) = \sqrt{\frac{2U}{k}}\frac{kV\cos\left(\sqrt{\frac{kU}{2}}t\right) - \sqrt{2kU}\sin\left(\sqrt{\frac{kU}{2}}t\right)}{\sqrt{2kU}\cos\left(\sqrt{\frac{kU}{2}}t\right) + kV\sin\left(\sqrt{\frac{kU}{2}}t\right)}. \tag{24}$$

The time period between t_2 and t_f is denoted as Δt_3, i.e., the duration of the third segment. Solving (24) for t with $x_2(0) = V$, yields the time Δt_3 when the velocity $x_2(t)$ reaches the final value x_{2f},

$$\Delta t_3 = \sqrt{\frac{2}{kU}}\tan^{-1}\Big(\frac{\sqrt{2k^3U}(V - x_{2f})}{2kU + x_{2f}k^2V}\Big). \tag{25}$$

Now, to obtain the net displacement Δx_3 traversed at this stage we need to integrate (24) from time $t_2 = 0$ to time $t_f = \Delta t_3$, giving,

$$\Delta x_3 = \frac{2}{k}\ln\Big(\frac{2kU + (Vk)^2}{\sqrt{\left(2kU + k^2Vx_{2f}\right)^2 + 2k^3U(V - x_{2f})^2}}\Big). \tag{26}$$

The last stage to compute is the middle segment, the constant velocity part of the motion where the velocity $x_2(t)$ is on the bound V. The displacement required for this stage is obtained from the difference of the initial and final conditions on x_1, and from the sum of Δx_1 and Δx_3,

$$\Delta x_2 = x_{1f} - x_{10} - \Delta x_1 - \Delta x_3. \tag{27}$$

Note that Δx_2 must be positive, so from that a condition for the system to reach the velocity bound can be obtained,

$$x_{1f} - x_{10} - \Delta x_1 - \Delta x_3 \geq 0. \tag{28}$$

If the condition in (28) is not satisfied, then the system does not reach the velocity bound, and the structure of this profile is discussed in the next section . Assuming that the condition in (28) is satisfied, then the system exhibit a constant velocity segment, thus during this segment $\dot{x}_2(t) = 0$, $u(t) = kV^2/2$ and the time interval for this stage is

$$\Delta t_2 = \frac{\Delta x_2}{V}. \tag{29}$$

The explicit expression for the velocity and displacement are $x_2(t) = V$ and $x_1(t) = Vt + \Delta x_1 + x_{10}$. The total value of the performance criterion and the time for the complete motion is

$$t_f = \Delta t_1 + \Delta t_2 + \Delta t_3. \tag{30}$$

Thus, we obtained the solution in the case where the velocity bound is reached. Also, necessary and sufficient conditions for the structure of the profile are derived, given in statement 1.

This solution was obtained according to the intelligent assumption that $p_2(t)$ is positive at first and monotonically decreasing, while zero when the profile is on the velocity bound. It yet has to be verified that this assumption does indeed hold. Specifically, $p_2(t)$ should be positive on the interval $[0, t_1]$, zero on the interval $[t_1, t_2]$ and negative on the interval $[t_2, t_f]$. Since there are analytical expressions for $x_2(t)$ we can solve (17) for $p_2(t)$. The first co-state $p_1(t)$ can be obtained from (15), $H(t) = 0$, $p_2(t_1) = 0$ and $x_2(t_1) = V$. Combining these facts results in $p_1(t) = 1/V$, and that $p_2(t)$ satisfies

$$\dot{p}_2(t) - kx_2(t)p_2(t) = -\frac{1}{V}. \tag{31}$$

This is a non-homogeneous ODE (Arnold 1973), and $\mu(t) = e^{-\int kx_2(s)ds}$ is the integration multiplier. The solution of (31) for $p_2(t), t \in [t_2, t_f]$ is

$$p_2(t) = -\frac{1}{\mu(t)}\Big(\int \mu(s)\frac{1}{V}ds + C_0\Big). \tag{32}$$

When looking at the third segment, i.e. when the velocity $x_2(t)$ is decreasing towards the final condition, we have that $x_2(t)$ is positive so the integral over it is increasing, and the integration multiplier $\mu(t)$ is positive and decreasing with time. Thus for $p_2(t)$ in (32) we can see that when the solution does not reside on the bound,$p_2(t)$ is a function of $-1/\mu(t)$ which is negative ($\mu(t)$ is positive), multiplied by $\int \mu(s)/V ds + C_0$ which is decreasing from zero value at $t = t_2$. Thus $p_2(t)$ is negative on $[t_2, t_f]$. The C_0 constant is such that at time t_2 we have $p_2(t_2) = 0$. On the first segment when $t \in [0, t_1]$ and the velocity is increasing towards the bound, we look at the reverse time equation of (31) and integrate it from zero on the bound at t_1 to zero, i.e. reverse time integration. The equation now becomes

$$\dot{p}_2(t) + kx_2(t)p_2(t) = \frac{1}{V} \tag{33}$$

and the integration multiplier is now $\mu(t) = e^{\int kx_2(s)ds}$, which is an increasing positive function. The explicit solution for p_2 now becomes,

$$p_2(t) = \frac{1}{\mu(t)}\Big(\int \mu(s)\frac{1}{V}ds + C_0\Big). \tag{34}$$

In (34), $p_2(t)$ is a multiplication of $1/\mu(t)$ which is positive, and $\int \mu(s)/V ds + C_0$ which is increasing from zero value at $t = t_1$. Therefore $p_2(t)$ is positive on $[t_1, 0]$ and increasing, so it is positive and decreasing on $[0, t_1]$. Here C_0 is such that $p_2(t_1) = 0$. It has been shown so far that during the acceleration segment p_2 is positive and decreasing while during the deceleration segment it is negative and decreasing. On the bound $p_2(t)$ is zero, and $\lambda_2(t) = 1/V$ in order to maintain $\dot{p}_2(t) = 0,\ t \in [t_1, t_2]$. This verifies that the structure of the co-states is indeed satisfied by the velocity trajectory that was calculated.

Theorem 1. *For a system described in* (12) *with the minimum time criterion in* (14)*, the solution is comprised of three segments with a Bang-Constant-Bang structure iff the following two conditions are satisfied:*

1. $V < \sqrt{\frac{2U}{k}}$

2. $\Delta x_a + \Delta x_d < x_{1f} - x_{10}$
$\Delta x_a = \frac{1}{k}\ln\Big(\frac{2kU-(x_{20}k)^2}{2kU-(Vk)^2}\Big)$,
$\Delta x_d = \frac{2}{k}\ln\Big(\frac{2kU+(Vk)^2}{\sqrt{(2kU+k^2Vx_{2f})^2+2k^3U(V-x_{2f})^2}}\Big)$

Otherwise the solution is comprised of two segments only, with a Bang-Bang structure.

Note, that if x_{20} (x_{2f}) is on the bound V, the solution collapses to a Constant-Bang (Bang-Constant) structure or even Constant all the way if $x_{2f} = x_{2f} = V$.

Non-active Upper Bound Velocity Constraint

If the conditions in statement 1 are not satisfied, then the solution does not reach the velocity bound, and the motion profile is comprised of two segments only where now $t_1 = t_2$ is the single switching point, i.e., an acceleration stage and a deceleration stage. We denote the maximum velocity reached during the acceleration stage as $\tilde{V}$ and the time of the switching of the control from $u(t) = U$ to $u(t) = -U$ as t_1. The expressions for $t_1, x_1(t_1)$ are the same as in the previous section but with the unknown velocity $x_2(t_1) = \tilde{V}$. The expression for the time of the switching point t_1 according to (21) is given by

$$t_1(\tilde{V}) = \frac{1}{\sqrt{2kU}}\ln\Big(\frac{\sqrt{2kU}-x_{20}}{\sqrt{2kU}+x_{20}}\frac{\sqrt{2kU}+\tilde{V}k}{\sqrt{2kU}-\tilde{V}k}\Big) \tag{35}$$

and the position according to (22) is

$$x_1(t_1,\tilde{V}) = x_{10} + \frac{1}{k}\ln\Big(\frac{2kU-(x_{20}k)^2}{2kU-(\tilde{V}k)^2}\Big). \tag{36}$$

Thus, the net displacement in the acceleration stage is

$$\Delta x_1(\tilde{V}) = \frac{1}{k}\ln\Big(\frac{2kU-(x_{20}k)^2}{2kU-(\tilde{V}k)^2}\Big). \tag{37}$$

The second stage is the deceleration stage where the system decelerate from the unknown maximum velocity reached $\tilde{V}$ at time $t = t_1$ to the final conditions at time $t = t_f$. The total time for this movement, i.e., $\Delta t_2 = t_f - t_1$ is according to (25),

$$\Delta t_2(\tilde{V}) = \sqrt{\frac{2}{kU}}\tan^{-1}\Big(\frac{\sqrt{2k^3U}(\tilde{V}-x_{2f})}{2kU+x_{2f}k^2\tilde{V}}\Big). \tag{38}$$

The expression for the net displacement in this stage is given be (26), with the unknown maximum velocity $\tilde{V}$,

$$\Delta x_2(\tilde{V}) = \frac{2}{k}\ln\Big(\frac{(\tilde{V}k)^2+2kU}{\sqrt{(2kU+k^2\tilde{V}x_{2f})^2+2k^3U(\tilde{V}-x_{2f})^2}}\Big). \tag{39}$$

In order to find $\tilde{V}$ it is required to solve the following equation for the $\tilde{V}$ parameter,

$$\Delta x_1(\tilde{V}) + \Delta x_2(\tilde{V}) = x_{1f} - x_{10}. \tag{40}$$

Equation (40) can be formulated after some mathematical smoothing into a quartic function in $\tilde{V}$ of the form,

$$a\tilde{V}^4 + b\tilde{V}^2 + c = 0 \tag{41}$$

where

$$\begin{aligned}
a &= (2k^3U)(1+\tilde{k}) + k^6(\tilde{k}x_{2f}^2 - x_{20}^2)\\
b &= (4k^3U)\big(2kU - (kx_{20})^2\big)\\
c &= (2kU)^2\big(2kU(1-\tilde{k}) - k^2(\tilde{k}x_{2f}^2 + x_{20}^2)\big)\\
\tilde{k} &= e^{k(x_{1f}-x_{10})}
\end{aligned}$$

Defining $V_m = \tilde{V}^2$, results in a quadratic function in V_m. Thus, (41) has four roots, described by

$$\tilde{V}_{1-4} = \pm\sqrt{\frac{-b\pm\sqrt{b^2-4ac}}{2a}}. \tag{42}$$

Since $\tilde{k} = e^{k(x_{1f}-x_{10})} \geq 1$ assuming $x_{1f} \geq x_{10}$ (when not the case the directions can be adjusted), it is easily observed that $c < 0,\ a > 0$. Thus the determinant must satisfy

$$4ac < 0 \quad \rightarrow \quad \sqrt{b^2-4ac} > \big|b\big| \tag{43}$$

Hence, there is only one positive real-valued root. Therefore the explicit solution for $\tilde{V}$ is given by

$$\tilde{V} = \sqrt{\frac{-b+\sqrt{b^2-4ac}}{2a}}. \tag{44}$$

The final time and value of the performance criterion in this case is

$$\begin{aligned}
t_f =& \frac{1}{\sqrt{2kU}}\ln\Big(\frac{\sqrt{2kU}-x_{20}}{\sqrt{2kU}+x_{20}}\frac{\sqrt{2kU}+\tilde{V}k}{\sqrt{2kU}-\tilde{V}k}\Big)\\
&+\sqrt{\frac{2}{kU}}\tan^{-1}\Big(\frac{\sqrt{2k^3U}(\tilde{V}-x_{2f})}{2kU+x_{2f}k^2\tilde{V}}\Big).
\end{aligned} \tag{45}$$

The optimality of this solution is proved identically as is done for Theorem 1 for the three segments case.

Empirical Evaluation

Solutions were obtained for the different cases, and compared to FALCON (Rieck et al. 2016). The drag coefficient is taken to be $k = 0.05$ in all the examples, representing a standard aircraft coefficient. In all simulations, three graphs are presented, the distance, velocity and control profiles as computed by the two algorithms, one on top of the other to show the agreement and difference between the two.

In Case 1 the velocity reaches the bound, and the conditions in statement 1 are satisfied for a three segments trajectory. The bounds were chosen as $V = 10$, $U = 10$, the initial condition $x_0 = [0,\ 1]^T$, and the final condition $x_f = [20,\ 4]^T$, see Fig. 1. The solution reached the velocity bound, and the structure was of a Bang-Constant-Bang profile. The motion time was 2.729 $[sec]$, and FALCON's calculation time was 19.21 $[sec]$. Note that on the singular arc, FALCON struggles with the singular control and oscillates around it.

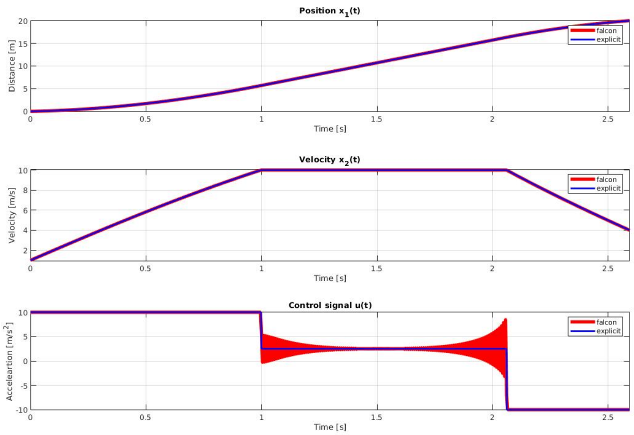

Figure 1: Three segments Bang-Constant-Bang profile with conditions satisfying the conditions in statement 1.

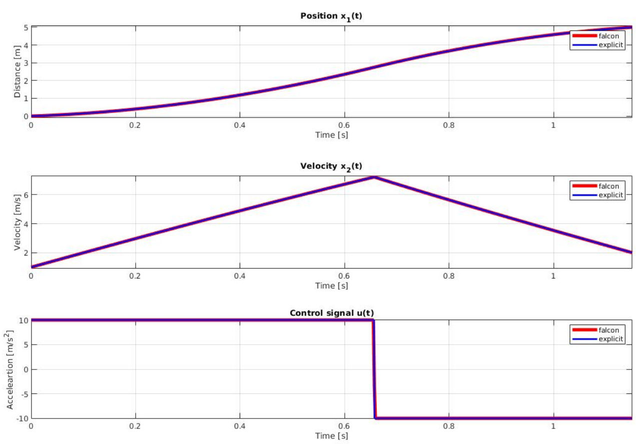

Figure 2: Two segments Bang-Bang profile, with second condition in statement 1 not being satisfied.

In Case 2 condition 2 in statement 1 is not satisfied, while condition 1 is satisfied. The bounds were chosen to be $V = 10$, $U = 10$, the initial condition $x_0 = [0,\ 1]^T$, and the final condition $x_f = [5,\ 2]^T$, see Fig. 2. The difference from the previous simulation is the distance to travel, the system did not have enough distance to accelerate in order achieve the velocity bound. The solution, in this case, exhibited a Bang-Bang profile. The motion time was 1.15 $[sec]$, and FALCON's calculation time was 10.4 $[sec]$.

For completeness, a third case is presented, where condition 1 in statement 1 is not satisfied, while the condition 2 is satisfied. The bounds were chosen to be $V = 20$, $U = 10$, the initial condition $x_0 = [0,\ 1]^T$, and the final condition was chosen to be far away (distance meaning) $x_f = [100,\ 4]^T$, see Fig. 3. It is clear that even though there is a very long distance to travel, the solution did not reach the bound, but asymptotically tended to it while never reaching it, and the maximal velocity attained was $\bar{V} = 19.87$.

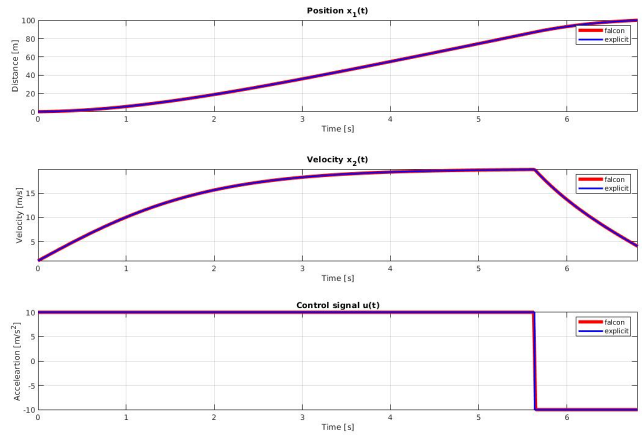

Figure 3: Two segments Bang-Bang profile with velocity approaching $V = \sqrt{\frac{2U}{k}}$.

Thus, the solution exhibits a Bang-Bang profile. The complete time for the motion was 6.8 $[sec]$, and FALCON's calculation time was 10.45 $[sec]$. Note that FALCON's time of motion was 6.97 $[sec]$ in this case, larger by 2.5% than the analytical time.

Conclusion and Future Work

We have addressed the problem of minimum time optimality for a second-order system with quadratic drag with control and velocity constraints which can be used to describe many general systems for the purpose of trajectory planning. The solution for the optimal time trajectory was found in analytical form, and necessary and sufficient conditions for the structure of the solution are also given.

These conditions are required in order to validate a given grounded plan. They also serve as tight lower bounds for the existence of any *solution* between an initial and final condition, as a function of the variables' start and end conditions at each event. We argue that finding these conditions is another step towards our long-term objective of producing a hybrid task planner that can find plans that are guaranteed to be physically feasible.

To achieve our long-term objective, we propose an iterative planning process, which is illustrated in Figure 4. First, we will use a planner to find a grounded plan based on a simple (first-order dynamical) model. Scotty is an ideal candidate for such a planner, but the scheme we propose below is agnostic to the choice of planner.

Then this grounded plan is validated with constraints that come from the more complex underlying dynamics of the problem. If all the constraints are satisfied, a trajectory can be computed for the complex model and returned as the solution for the problem. If the constraints (or a subset of them) are not satisfied, the problem should be modified to eliminate that solution and the whole process can be repeated for the modified problem. This final part is future work.

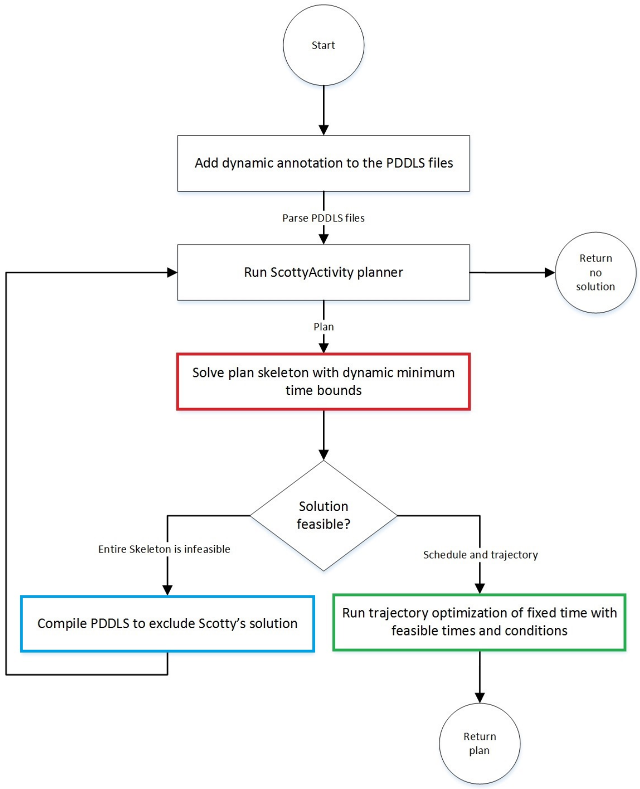

Figure 4: Flow of the hybrid planning algorithm combining optimal control and Scotty.

References

Arnold, V. I. 1973. *Ordinary Differential Equations*. London: The MIT Press.

Ata, A. A. 2007. Optimal trajectory planning of manipulators: a review. *Journal of Engineering Science and Technology* 2(1):32–54.

Berger, A.; Ioslovich, I.; and Gutman, P.-O. 2015. Time optimal trajectory planning with feedforward and friction compensation. In *American Control Conference (ACC), 2015*, 4143–4148. IEEE.

Bittanti, S.; Laub, A. J.; and Willems, J. C. 2012. *The Riccati Equation*. Springer Science & Business Media.

Coles, A. J.; Coles, A. I.; Fox, M.; and Long, D. 2012. Colin: Planning with continuous linear numeric change. *Journal of Artificial Intelligence Research* 44:1–96.

Fernandez-Gonzalez, E.; Williams, B.; and Karpas, E. 2018. Scottyactivity: Mixed discrete-continuous planning with convex optimization. *Journal of Artificial Intelligence Research* 62:579–664.

Fox, M., and Long, D. 2003. Pddl2. 1: An extension to pddl for expressing temporal planning domains. *Journal of artificial intelligence research* 20:61–124.

Hartl, R. F.; Sethi, S. P.; and Vickson, R. G. 1995. A survey of the maximum principles for optimal control problems with state constraints. *SIAM review* 37(2):181–218.

Kirk, D. E. 2012. *Optimal control theory: an introduction*. New York: Courier Corporation.

Li, H. X., and Williams, B. C. 2008. Generative planning for hybrid systems based on flow tubes. In *ICAPS*, 206–213.

Pontryagin, L.; Boltyanskii, V.; Gamkrelidze, R.; and Mishechenko, E. 1962. *Mathematical theory of optimal processes*. New York/London: John Wiley & Sons.

Rieck, M.; Bittner, M.; Gruter, B.; and Diepolder, J. 2016. Falcon.m. http://www.fsd.mw.tum.de/software/falcon-m.

Taitler, A.; Ioslovich, I.; Gutman, P.-O.; Karpas, E.; and Ben-Asher, J. Z. 2019a. Structural Analysis Of Combined Time and Energy Optimal Trajectory Planning With Quadratic Drag. In *2019 27th Mediterranean Conference on Control and Automation (MED) - submitted*. IEEE.

Taitler, A.; Ioslovich, I.; Gutman, P.-O.; and Karpas, E. 2019b. Combined time and energy optimal trajectory planning with quadratic drag for mixed discrete-continuous task planning. *Optimization* 68(1):125–143.

Verscheure, D.; Demeulenaere, B.; Swevers, J.; De Schutter, J.; and Diehl, M. 2008. Time-energy optimal path tracking for robots: a numerically efficient optimization approach. In *Advanced Motion Control, 2008. AMC'08. 10th IEEE International Workshop on*, 727–732. IEEE.

Wächter, A., and Biegler, L. T. 2006. On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming. *Mathematical programming* 106(1):25–57.